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Technical Note

The stability of flow in a channel or duct occupied by a porous medium

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Abstract

The hydrodynamic stability of flow of an incompressible fluid through a plane-parallel channel or circular duct filled with a saturated porous medium, modeled by the Brinkman equation, is discussed on the basis of an analogy with a magneto-hydrodynamic problem (Hartmann flow). Flow in a circular duct is found to be stable to small disturbances for all values of the Reynolds number, but for a plane-parallel channel the flow is unstable if the Reynolds number exceeds a critical value, dependent on a Darcy number.

Keywords: Porous medium; Stability; Channel; Duct

1. Introduction

Until recently little interest was taken in turbulence in porous media on a macroscopic scale (on the scale of a representative elementary volume), as distinct from turbulence within the pores. However, with the advent of hyperporous materials (such as the metallic foams used in the cooling of electronic equipment) there has been a substantial increase in interest in this topic. The subject has been surveyed by Lage et al. [1].

As far as the author is aware, there has been no discussion in the literature about the stability problem relating to the onset of macroscopic turbulence. Indeed, if one limits oneself to flow modeled by Darcy's law there is little to discuss. The flow in a duct or channel is then simply a slug flow, and it is well known that this is stable, to small disturbances at least, for all values of the Reynolds number. Indeed, in this case one can apply a Galilean transform and find a reference frame in which the fluid is at rest. On the other hand, if the porosity is very high then one would expect to have a situation close to the flow of a Navier–Stokes fluid through a duct or channel. It is well known that Poiseuille flow in a circular duct is stable to small disturbances at all values of the Reynolds number, but for the case of plane Poiseuille flow the flow becomes unstable to small disturbances when the Reynolds number (based on half the channel width) exceeds the critical value 5772 (see, for example, [2]). Thus in the case of flow in a plane-parallel channel filled with a hyperporous material, one would expect there to be a critical Reynolds number, dependent on the Darcy number *Da*, taking a value close to 5772 for very large values of *Da*, and increasing as *Da* decreases as a result of the velocity profile flattening due to the presence of the solid matrix of the porous medium. On the other hand, one would expect that in the case of a circular duct filled with porous material, the flow would be stable to small disturbances no matter what are the values of the Reynolds number and the Darcy number.

In this paper the stability of flow in the plane-parallel channel situation is examined in detail. Normally one would have to spend a great deal of effort solving the Orr–Sommerfeld equation for the appropriate class of velocity profiles. Fortunately, one can here avoid this effort by making use of an analogy between the flow in

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a channel filled with a porous medium modeled by the Brinkman equation, and a well-known type of magnetohydrodynamic (MHD) flow, namely Hartmann flow [3]. In Section 2 this analogy is established in detail, and in Section 3 the analogy is applied.

2. The analogy between Hartmann flow and flow in a porous medium

Under the usual MHD approximations, flow of an electrically conducting fluid is governed by the momentum equation obtained by adding a term representing force per unit volume, equal to $\mathbf{j} \times \mathbf{B}$, to the right-hand side of the Navier–Stokes equation, where $\mathbf{B}$ is the magnetic induction and $\mathbf{j}$ is the current density, given in the absence of an applied electric field by

$$\mathbf{j} = \sigma(\mathbf{v} \times \mathbf{B}). \tag{1}$$

Here σ is the electrical conductivity and $\mathbf{v}$ is the fluid velocity vector. We now consider the case where the magnetic diffusivity is very large, so that perturbations from an applied magnetic induction $\mathbf{B}_0$ die away rapidly.

Suppose that one has a Cartesian frame in which the x-axis is directed along the axis of the channel and the z-axis is directed in the transverse direction. Then in the case of axial flow and a uniform transverse applied magnetic field one has

$$\mathbf{v} = (u(z), 0, 0), \quad \mathbf{B} = (0, 0, B_0), \quad \text{and so}$$
$$\mathbf{j} = (0, -\sigma B_0 u(z), 0), \quad \mathbf{j} \times \mathbf{B} = (-\sigma B_0^2 u(z), 0, 0). \tag{2}$$

In the case of steady flow, the x-component of the momentum equation is

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} - \sigma B_0^2 u. \tag{3}$$

For the case of constant pressure gradient $G = -\mathrm{d}p/\mathrm{d}x$, the solution of Eq. (3) subject to the no-slip boundary conditions

$$u = 0 \quad \text{at } z = \pm L \tag{4}$$

is

$$u = U_0 \frac{\cosh M - \cosh(Mz/L)}{\cosh M - 1}, \tag{5}$$

where U_0 is the velocity at the center line of the channel and

$$M = \left(\frac{\sigma}{\mu}\right)^{1/2} B_0 L. \tag{6}$$

The dimensionless parameter M is commonly called the Hartmann number.

Eq. (3) may be compared with the corresponding equation for flow in a porous medium governed by the Brinkman equation, namely

$$0 = -\frac{\mathrm{d}p}{\mathrm{d}x} + \mu \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} - \frac{\phi\mu}{K} u. \tag{7}$$

Here u denotes the intrinsic velocity, related to the Darcy velocity u_D by the Dupuit–Forchheimer relationship $u_\mathrm{D} = \phi u$, and it has been assumed that the effective viscosity is equal to μ/ϕ. It is clear that Eqs. (3) and (7) become identical if one identifies σB_0^2 with $\phi\mu/K$. This is equivalent to identifying the Hartmann number M with $(\phi/Da)^{1/2}$, where Da is the Darcy number defined by $Da = K/L^2$. Further, the applicable boundary conditions are the usual hydrodynamic ones in each case. (In the case of large magnetic diffusivity the electromagnetic boundary conditions do not affect the hydrodynamic stability problem.) Thus there is an exact analogy between the MHD problem and the Brinkman porous medium problem as far as the basic velocity profile is concerned.

In order to discuss the hydrodynamic stability problem, we need to include the inertial terms in the momentum equation and extend the equation from one to two dimensions. (Squire's theorem provides justification for working in two dimensions rather than three dimensions.) We will assume that in the porous medium the Brinkman equation takes form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u} - \frac{\phi\mu}{K}\mathbf{u}. \tag{8}$$

For two-dimensional flow, $\mathbf{u} = (u, 0, w)$. The reader will note that the Forchheimer quadratic drag term has been omitted. The expected effect on the stability resulting from the inclusion of this term is discussed below.

The full form of the momentum equation for the MHD problem is

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = -\nabla p + \mu \nabla^2 \mathbf{u} - \sigma B_0^2 (u, 0, 0). \tag{9}$$

The analogy is no longer exact, because in the MHD problem the transverse component of the velocity has no effect on the drag term (since it is parallel to the applied magnetic field). However, the author would argue that the effect of this difference on the linear instability problem, in which the perturbations (and in particular the transverse velocity) are assumed to be infinitesimal, is expected to be small. In fact, the detailed analysis described below shows that the effect of the difference is zero.

The non-dimensional form of Eq. (8), based on length scale L, velocity scale U_0, and time scale L/U_0 is

$$R\left(\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}}\right) = -\nabla \hat{p} + \hat{\nabla}^2 \hat{\mathbf{u}} - M_\mathrm{D}^2 \hat{\mathbf{u}}, \tag{10}$$

where R is the Reynolds number, defined by

$$R = \rho U_0 L/\mu, \tag{11}$$

and M_D denotes the Darcy analogue of the Hartmann number, defined by

$$M_D = \left(\frac{\phi}{Da}\right)^{1/2}. \tag{12}$$

The circumflexes in Eq. (10) are now dropped, so that they can be used again in another context. The perturbation analysis now follows the familiar path, as described in Drazin and Reid [2].

Small perturbations from the basic steady-state solution are denoted by primes. We let

$$\mathbf{u}(\mathbf{x},t) = U(z)\mathbf{i} + \mathbf{u}'(\mathbf{x},t),\quad p(\mathbf{x},t) = P(z) + p'(\mathbf{x},t), \tag{13}$$

where $U(z)$ and $P(z)$ refer to the basic solution. We substitute in Eq. (10) and neglect second-order terms. The linearized forms of (10) and the continuity equation are

$$R\left\{\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\mathbf{u}' + w'\frac{\mathrm{d}U}{\mathrm{d}z}\right\} = -\nabla p' + \hat{\nabla}^2\mathbf{u}' - M_D^2\mathbf{u}' \quad \text{and} \quad \nabla\cdot\mathbf{u}' = 0. \tag{14}$$

The normal mode analysis involves the introduction of solutions of the form

$$\mathbf{u}'(\mathbf{x},t) = \hat{\mathbf{u}}(z)\exp[\mathrm{i}(\alpha x - \alpha ct)], \tag{15}$$

$$p'(\mathbf{x},t) = \hat{p}(z)\exp[\mathrm{i}(\alpha x - \alpha ct)]. \tag{16}$$

Substitution into Eq. (14) yields

$$\{D^2 - \alpha^2 - M_D^2 - \mathrm{i}\alpha R(U-c)\}\hat{u} = RU'\hat{w} + \mathrm{i}\alpha\hat{p},\quad \{D^2 - \alpha^2 - \mathrm{i}\alpha R(U-c)\}\hat{w} = D\hat{p},\quad \mathrm{i}\alpha\hat{u} + D\hat{w} = 0. \tag{17a,b,c}$$

Elimination of $\hat{u}$ and $\hat{p}$ in turn gives

$$\frac{1}{\mathrm{i}\alpha R}(D^2-\alpha^2)^2\hat{w} = (U-c)(D^2-\alpha^2)\hat{w} - U''w + \frac{\mathrm{i}M_D^2}{\alpha R}D^2\hat{w}. \tag{18}$$

This is the Orr–Sommerfeld equation for the porous medium problem. It has to be solved subject to the appropriate boundary conditions. The no-slip condition (together with the equation of continuity) implies that

$$\hat{w} = D\hat{w} = 0 \quad \text{at } z = 1 \text{ and at } z = -1. \tag{19}$$

Allowing for differences in notation, Eq. (18) is identical with Eq. (26) in a paper by Lock [4]. Lock comments that this equation differs from the Orr–Sommerfeld equation of ordinary hydrodynamics only in the last term and this is negligible for either neutral or amplified oscillations, and that in the Hartmann problem the principle effect of the magnetic field is the modification of the basic velocity distribution. Amongst other things, Lock establishes the appropriate analogue of Squire's theorem for the present problem.

A similar analogy, but one applicable to a natural convection situation, was observed by Vasseur et al. [5].

3. Application of the analogy

Thus there is no need to perform new calculations for the porous medium problem. One can quote those results tabulated by Lock [4], or even better the more accurate results computed by Takashima [6]. The latter are presented in Table 1.

4. Discussion

As one would expect, the effect of an insertion of a porous solid matrix into a channel is to stabilize the flow. Flow in a circular tube is already stable to small disturbances even for the case of a fluid clear of solid material, so it is expected to stay stable to small disturbances when the solid matrix is added.

The case a plane-parallel channel is more interesting. The results presented here show that the flow will become unstable to small disturbances at a sufficiently high Reynolds number, but when the Darcy number is small the critical value of the Reynolds number is very high. The appearance of instability does not necessarily imply the onset of turbulence, since it is possible, and indeed likely, that the wavy disturbances will grow in a more or less regular fashion in the case of a porous medium with moderate or small Darcy number. The

Table 1
The effect of the parameter $M_D = (\phi/Da)^{1/2} = (\phi L^2/K)^{1/2}$ on the critical Reynolds number R_c, the critical wavenumber α_c, and the critical wave speed c_c

M_D	R_c	α_c	c_c
0.0	5772	1.0205	0.2640
0.5	6706	1.0057	0.2559
1.0	10016	0.9718	0.2355
2.0	28604	0.9278	0.1921
3.0	65155	0.9582	0.1690
4.0	112395	1.0355	0.1598
5.0	164090	1.1342	0.1564
10.0	439818	1.7391	0.1548
15.0	708962	2.4566	0.1550
20.0	961767	3.2376	0.1550
30.0	1449060	4.8461	0.1550
50.0	2415550	8.0766	0.1550
70.0	3381771	11.3072	0.1550
100.0	4831101	16.1531	0.1550
200.0	9662203	32.3063	0.1550

Values based on Table 1 of [6].

problem of the stability to disturbances of finite amplitude, at Reynolds numbers subcritical with respect to the linear stability problem, remains to be investigated. One would expect that the onset of this sort of instability, if it occurs, would be associated with a transition to macroscopic turbulence.

The addition of a Forchheimer quadratic drag term to the momentum equation would be expected to further stabilize the flow, primarily as a result of a flattening of the basic velocity profile. Also, for the case of a given applied pressure gradient, the increased drag would result in a reduced mean velocity, and that would make it harder for any critical Reynolds number to be reached. This means that the critical Reynolds numbers given in Table 1 should be regarded as lower bounds on the true critical Reynolds number in an experimental situation.

In fact, a quantitative estimate of the effect of quadratic (form) drag can be obtained as follows. The argument below is based on the assumption that the stability criterion is determined primarily by the shape of the velocity profile, together with the observation that that shape is determined by the ratio of coefficients of terms in the momentum equation made quasi-linear in the velocity. The Forchheimer extension involves the addition of the term

- $(c_F\rho/K^{1/2})\phi^2|\mathbf{u}|\mathbf{u}$ to the right-hand side of Eq. (8), and hence a term
- $(c_F L/K^{1/2})R\phi^2|\hat{\mathbf{u}}|\hat{u}$ to the right-hand side of Eq. (9). If one approximates $|\hat{\mathbf{u}}|$ by 1, then one has Eq. (9) with M_D replaced by an effective value M_D^* where

$$M_D^* = \left(\frac{\phi}{Da}[1 + c_F\phi Da^{1/2}R]\right)^{1/2}. \quad (20)$$

The Reynolds number R can now be approximated by R_c, given by Table 1, corresponding to the value of M appropriate for the case $c_F = 0$. With M_D^* thus obtained, one can go back to Table 1 and read off a new value of the critical Reynolds number R_c. (If one wants to, one can iterate the process.) It is clear that the Forchheimer effect leads to an increase in the effective value of M_D, and hence to an increase in the critical Reynolds number, as expected.

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